Problem 1

a) $-1+i$

$$r=\sqrt{(-1)^2+1^2}$$
$$=\sqrt{2}$$

$$\theta = \tan^{-1}\frac{1}{-1}$$
$$=\pi-\frac{\pi}{4}$$
$$=\frac{3\pi}{4}$$

$$z=\sqrt{2}e^{\frac{3\pi}{4}i}$$

b) $\sqrt{3}-i$

$$r=\sqrt{\sqrt{3}^2+(-1)^2}$$
$$=2$$

$$\theta = \tan^{-1}\frac{-1}{\sqrt{3}}$$
$$=\frac{\pi}{6}$$

$$z=2e^{\frac{\pi}{6}i}$$

Problem 2

$$\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)}$$

$$= \frac{1-2i+i^2}{1-i^2}$$

$$= \frac{-2i}{2}$$

$$= -i$$

$z_1 = 1-i$

$r_1 = \sqrt{2}$

$\theta_1 = \tan^{-1}\frac{-1}{1}$

$= 2\pi - \frac{\pi}{4}$

$= \frac{7\pi}{4}$

$z_2 = 1+i$

$r_2 = r_1$

$= \sqrt{2}$

$\theta_2 = \frac{\pi}{4}$

$$z = \frac{z_1}{z_2}$$

$$= \frac{\sqrt{2}e^{\frac{7\pi}{4}i}}{\sqrt{2}e^{\frac{\pi}{4}i}}$$

$$= e^{\frac{6\pi}{4}i}$$

$$= e^{\frac{3\pi}{2}i}$$

$$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 + i(-1)$$

$$= -i$$

Problem 3

a) $(1-i)^4$

$$= \binom{4}{0} 1^4 \cdot (-i)^0 + \binom{4}{1} 1^3 \cdot (-i)^1 + \binom{4}{2} 1^2 \cdot (-i)^2$$

$$+ \binom{4}{3} 1 \cdot (-i)^3 + \binom{4}{4} 1^0 \cdot (-i)^4$$

$$= 1 - 4i + 6i^2 - 4i^3 + i^4$$

$$= 1 - 4i - 6 + 4i + 1$$

$$= -4$$

$(1-i)^4$ $\qquad r = \sqrt{2}$

$= \left(\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\right)^4$ $\qquad \theta = \tan^{-1}\frac{-1}{1}$

$\qquad = \frac{7\pi}{4}$

$$= 4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^4$$

$$= 4\left(\cos 7\pi + i\sin 7\pi\right)$$

$$= 4\left(-1 + 0\right)$$

$$= -4$$

b) $(1+i\sqrt{3})^3$

$$= \binom{3}{0} 1^3 \cdot i\sqrt{3}^0 + \binom{3}{1} 1^2 \cdot i\sqrt{3}^1 + \binom{3}{2} 1^1 \cdot i\sqrt{3}^2$$

$$+ \binom{3}{3} 1^0 \cdot i\sqrt{3}^3$$

$$= 1 + 3\sqrt{3}i + 9i^2 + 3\sqrt{3}i^3$$

$$= 1 + 3\sqrt{3}i - 9 - 3\sqrt{3}i$$

$$= -8$$

c) $(1+i\sqrt{3})^3$

$$= 2^3 \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3$$

$$= 8\left(\cos\pi + i\sin\pi\right)$$

$$= 8(-1+0)$$

$$= -8$$

Problem 4

$\sqrt[6]{1}$

Let $z = \sqrt[6]{1}$, then $z^6 = 1$.

$$r^6 e^{i6\theta} = 1 \cdot e^{2k\pi i}$$

$$r = 1, \quad 6\theta = 2k\pi$$

$$\theta = \frac{2k\pi}{6}, \quad k = 0, 1, \dots, 5$$

$\sqrt[6]{1} = z_0, z_0\zeta, \dots, z_0\zeta^5$, where $z_0 = \sqrt[n]{1} e^{\frac{i\theta}{n}}$

$$z_0 = e^0$$

$$z_1 = e^{\frac{2\pi}{6}i} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2 = e^{\frac{4\pi}{6}i} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_3 = e^{\frac{6\pi}{6}i} = \cos\pi + i\sin\pi = -1$$

$$z_4 = e^{\frac{8\pi}{6}i} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z_5 = e^{\frac{10\pi}{6}i} = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Problem 5

$$x^4 + 16 = 0 \qquad \text{Let } z^4 = -16.$$

$$x^4 = -16$$

$$x^2 = 4i, \quad x^2 = -4i$$

$$= 4e^{\frac{\pi}{2}i}, \quad = 4e^{-\frac{\pi}{2}i}$$

$$x_1 = \sqrt{4}\, e^{\frac{\pi}{4}i} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} + i\sqrt{2}$$

$$x_2 = -\sqrt{4}\, e^{\frac{\pi}{4}i} = \sqrt{2} - i\sqrt{2}$$

$$x_3 = \sqrt{4}\, e^{-\frac{\pi}{4}i} = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} - i\sqrt{2}$$

$$x_4 = -\sqrt{4}\, e^{-\frac{\pi}{4}i} = -\sqrt{2} + i\sqrt{2}$$